

DIRAC OSCILLATOR VIA R-DEFORMED HEISENBERG ALGEBRA

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Abstract

The complete energy spectrum for the Dirac oscillator via R-deformed Heisenberg algebra is investigated.

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I. INTRODUCTION

The relativistic Dirac oscillator proposed by Moshinsky-Szczepaniak [1] is a spin $\frac{1}{2}$ object with the Hamiltonian which in the non-relativistic limit leads to that of a 3-dimensional isotropic oscillator shifted by a constant term plus a $\vec{L} \cdot \vec{S}$ coupling term for both signs of energy. There they construct a Dirac Hamiltonian, linear in the momentum \vec{p} and position \vec{r} , whose square leads to the ordinary harmonic oscillator in the non-relativistic limit. The Dirac oscillator have been investigated in several context [2–8]

The R-deformed Heisenberg algebra or Wigner-Heisenberg algebraic technique [9] was recently super-realized for the SUSY isotonic oscillator [17,18]. The R-Heisenberg algebra has also been investigated for the three-dimensional non-canonical oscillator to generate a representation of the orthosymplectic Lie superalgebra $osp(3/2)$ [12].

The R-Heisenberg algebra has been found relevant in the context of integrable models [13], and the Calogero interaction [14,15]. Recently it has been employed for bosonization of supersymmetry in quantum mechanics [16], and the discrete space structure for the 3D

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Wigner quantum oscillator has been investigated [20]. In this work, we obtain the complete energy spectrum for the Dirac oscillator via R-deformed Heisenberg (RDH) algebra.

II. 3D WIGNER OSCILLATOR

In this Section, we provide a three dimension presentation of the Wigner system with its bosonic sector to be the 3D isotropic oscillator (assumed to be of spin- $\frac{1}{2}$, to aid factorization).

The R-deformed Heisenberg (or Wigner-Heisenberg) algebra is given by following (anti-)commutation relations ($[A, B]_+ \equiv AB + BA$ and $[A, B]_- \equiv AB - BA$) :

$$H = \frac{1}{2}[a^-, a^+]_+, \quad [H, a^\pm]_- = \pm a^\pm, \quad [a^-, a^+]_- = 1 + cR, \quad [R, a^\pm]_+ = 0, \quad R^2 = 1, \quad (1)$$

where c is a real constant associated to the Wigner parameter [17]. Note that when $c = 0$ we have the standard Heisenberg algebra.

It is straightforward, following the analogy with the Ref. [17], to define the super-realizations for the ladder operators $a^\mp(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$ for $H_W \equiv H(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$ taking the explicitly forms

$$a^\mp = a^\mp(\vec{\sigma} \cdot \vec{L} + \mathbf{1}) = \frac{1}{\sqrt{2}} \left\{ \mp \Sigma_1 \left(\frac{\partial}{\partial r} + \frac{1}{r} \right) \pm \frac{1}{r} (\vec{\sigma} \cdot \vec{L} + \mathbf{1}) \Sigma_1 \Sigma_3 - \Sigma_1 r \right\} \quad (2)$$

which satisfy together with $H_W \equiv H(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$ all the algebraic relations of the RDH algebra with the constant $\frac{c}{2}$ replaced by $(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$ and $R = \Sigma_3$. Note that $(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$ commutes with all the basic elements (a^\mp and H_W) of the RDH algebra.

It may be observed that the RDH algebra that gets defined here is in fact three dimensional (one dimension for r and two for $(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$) and is identically satisfied on any arbitrary three dimensional wave function.

On the eigenspaces of the operator $(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$, the 3D Wigner algebra gets reduced to a 1D from with $(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$ replaced by its eigenvalue $\mp(\ell + 1)$, $\ell = 0, 1, 2, \dots$, where ℓ is the orbital angular momentum quantum number. The eigenfuncitons of $(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$ for the eigenvalues $(\ell + 1)$ and $-(\ell + 1)$ are respectively given by the well known spin-spherical harmonic y_\mp .

Now, considering simultaneous eigenfuncitons of the mutually commuting H_W and $(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$ by

$$\psi_{W,+} = \begin{pmatrix} \tilde{R}_{1,+}(r) \\ \tilde{R}_{2,+}(r) \end{pmatrix} y_+, \quad (\vec{\sigma} \cdot \vec{L} + \mathbf{1})\psi_{W,+} = (\ell + 1)\psi_{W,+}, \quad (3)$$

and

$$\psi_{W,-} = \begin{pmatrix} \tilde{R}_{1,-}(r) \\ \tilde{R}_{2,-}(r) \end{pmatrix} y_-, \quad (\vec{\sigma} \cdot \vec{L} + \mathbf{1})\psi_{W,-} = -(\ell + 1)\psi_{W,-}, \quad (4)$$

(where the use of the subscript $+$ ($-$) indicates association with $[y_+(y_-)]$, we observe that the positive semi-definite form of H_W the ladder relations and the form of H_W dictat that the ground state energy $E_w^{(0)}(\vec{\sigma} \cdot \vec{L} + \mathbf{1}) \geq 0$, where $E_W(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$ indicates a function of $\vec{\sigma} \cdot \vec{L} + \mathbf{1}$, is determined by the annihilation condition which reads as two cases.

III. THE DIRAC OSCILLATOR MODEL

Adding an "anomalous momentum" in the form of a (nonlocal) linear and hermitian interaction,

$$\vec{\alpha} \cdot \vec{\pi} \equiv -iM\omega\beta\vec{\alpha} \cdot \vec{r} = (\vec{\alpha} \cdot \vec{\pi})^\dagger, \quad (5)$$

in the (noncovariant) Dirac free particle equation with mass M and spin- $\frac{1}{2}$, in the natural sistem of units,

$$i\frac{\partial\psi}{\partial t} = (\vec{\alpha} \cdot \vec{p} + M\beta)\psi, \quad (6)$$

one obtains the equation for the Dirac oscillator [1]:

$$i\frac{\partial\psi}{\partial t} = \{\vec{\alpha} \cdot (\vec{p} + \vec{\pi}) + M\beta\}\psi, \quad (7)$$

where M and ω are, respectively, the mass of the particle and the frequency of the oscillator, and the matrices $(\vec{\alpha}, \beta)$ satisfy the following properties:

$$[\alpha_i, \beta]_+ = 0, \quad [\alpha_i, \alpha_j]_+ = 2\delta_{ij}\mathbf{1}, \quad \beta^2 = \mathbf{1} = \alpha_i^2, \quad (i, j = 1, 2, 3). \quad (8)$$

Writing the Dirac spinor in terms of the upper and lower components, respectively, ψ_1 and ψ_2 ,

$$\Psi(\vec{r}, t) = \exp(-iEt)\Psi(\vec{r}), \quad \Psi(\vec{r}) = \begin{bmatrix} \psi_1(\vec{r}) \\ \psi_2(\vec{r}) \end{bmatrix} \quad (9)$$

the standard representation of the matrices $\vec{\alpha}$ and β .

IV. THE DIRAC OSCILLATOR VIA RDH ALGEBRA

In this section, we implement a new realization of the Dirac oscillator in terms of elements of the R-deformed Heisenberg algebra. To solve the equation Dirac, following the usual procedure, we consider the second order differential equation,

$$\tilde{H}_D\psi(\vec{r}) = E\psi(\vec{r}), \quad (10)$$

where \tilde{H}_D is a second order Hamiltonian,

$$\tilde{H}_D = H_D^2 + M^2\mathbf{1}, \quad \tilde{E} = \frac{E^2 - M^2}{2M}. \quad (11)$$

In the spherical polar coordinate system, we obtain the non-relativistic form of the Hamiltonian U [19], for an isotropic 3D SUSY harmonic oscillator with spin- $\frac{1}{2}$.

We consider a unitary operator in terms of the radial projection of the spin,

$$U = \begin{bmatrix} 1 & 0 \\ 0 & \sigma_r \end{bmatrix} = U^{-1} = U^\dagger, \quad (12)$$

to obtain the following relation between the transformed Dirac Hamiltonian, \tilde{H}_D , the 3D Wigner Hamiltonian, H_W , and the SUSY Hamiltonian, H_{SUSY} [19]:

$$H_{\text{SUSY}} = U\tilde{H}_DU^\dagger = H_W - \frac{1}{2}\{1 + 2(\sigma \cdot \vec{L} + \mathbf{1})\Sigma_3\}\omega\Sigma_3. \quad (13)$$

A. The energy spectrum of the Dirac oscillator

The energy spectra of the operators \tilde{H}_D and H_{SUSY} are identical, since these operators are related by a unitary transformation. However, the relation between the principal quantum number N and the angular momentum (ℓ) is different, in each case. Obviously, the energy spectrum associated with the two types of eigenspaces belonging to the eigenvalues $\pm(\ell + 1)$:

$$\text{Case(i)} \rightarrow \vec{\sigma} \cdot \vec{L} + \mathbf{1} \rightarrow \ell + 1 = j + \frac{1}{2}, \quad j = \ell + \frac{1}{2}$$

$$\tilde{E}_{N\ell} = \frac{E^2 - M^2}{2M} = \begin{cases} 2m\omega = \tilde{E}_{N(\ell+1)}^+, \\ 2(m+1)\omega = \tilde{E}_{N\ell}^-, \end{cases} \quad (14)$$

where $m = 0, 1, 2, \dots$

$$\text{Case(ii)} \rightarrow \vec{\sigma} \cdot \vec{L} + \mathbf{1} \rightarrow -(\ell + 1) = -(j + \frac{1}{2}), \quad j = (\ell + 1) - \frac{1}{2}$$

$$\begin{aligned} \tilde{E}_{N\ell} &= \frac{E^2 - M^2}{2M} \\ &= \begin{cases} (N + j + 3/2)\omega = \tilde{E}_{N\ell}^+, & N = j - \frac{1}{2}, j + 3/2, j + 7/2, \dots, \\ (N + j + 5/2)\omega = \tilde{E}_{N(\ell+1)}^-, & N = j + \frac{1}{2}, j + 5/2, \dots. \end{cases} \end{aligned} \quad (15)$$

V. CONCLUSION

In this work we investigate an interesting quantum system, the so-called Dirac oscillator, first introduced by Moshinsky-Szczepaniak [1]; its spectral resolution has been investigated with the help of techniques of super-realization of the R-deformed Heisenberg algebra.

The Dirac oscillator with different interactions has been treated by Castaños *et al.* [7] and by Dixit *et al.* [8]. These works motivate the construction of a new linear Hamiltonian in terms of the momentum, position and mass coordinates, through a set of seven mutually anticommuting 8x8-matrices yielding a representation of the Clifford algebra $C\ell_7$. The seven elements of the Clifford algebra $C\ell_7$ generate the three linear momentum components, the three position coordinates components and the mass, and their squares are the 8x8-identity matrix $\mathbf{I}_{8 \times 8}$. Results of our analysis on Dirac oscillator via the Clifford algebra $C\ell_7$ are in preparation.

In a forthcoming paper we show that the Dirac oscillator equation can be resolved algebraically without having to transform it into a second order differential equation. Therefore, the important connection for the Dirac 3D-isotropic oscillator with the linear ladder operators of the R-deformed Heisenberg algebra, satisfying the concomitant general oscillator quantum rule of Wigner, have explicated in this work.

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